

# Unmixed bipartite graphs

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## Abstract

In this note we give a combinatorial characterization of all the unmixed bipartite graphs.

## Resumen

En esta nota nosotros presentamos una caracterización combinatoria de todas las gráficas bipartitas no-mezcladas.

## 1 Unmixed graphs

In the sequel we use [3] as a reference for standard terminology and notation on graph theory.

Let  $G$  be a simple graph with vertex set  $V(G)$  and edge set  $E(G)$ . A subset  $C \subset V(G)$  is a *minimal vertex cover* of  $G$  if: (1) every edge of  $G$  is incident with one vertex in  $C$ , and (2) there is no proper subset of  $C$  with the first property. If  $C$  satisfies condition (1) only, then  $C$  is called a *vertex cover* of  $G$ . Notice that  $C$  is a minimal vertex cover if and only if  $V(G) \setminus C$  is a maximal independent set. A graph  $G$  is called *unmixed* if all the minimal vertex covers of  $G$  have the same number of elements and it is called *well covered* [6] if all the maximal independent sets of  $G$  have the same number of elements.

The notion of unmixed graph is related to some other graph theoretical and algebraic properties. The following implications hold for any graph without isolated vertices [1, 3, 8]:

$$\text{Cohen-Macaulay} \implies \text{unmixed} \implies B\text{-graph} \implies \text{vertex-critical}.$$

Structural aspects of Cohen-Macaulay bipartite graphs were first studied in [2]. In loc. cit. it is shown that  $G$  is Cohen-Macaulay if and only if the simplicial complex

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$\Delta_G$  generated by the maximal independent sets of  $G$  is shellable. The main result that we present in this note is the following combinatorial characterization of all the unmixed bipartite graphs. Our result is inspired by a criterion of Herzog and Hibi [4, Theorem 3.4] that describe all Cohen-Macaulay bipartite graphs in combinatorial terms.

**Theorem 1.1** *Let  $G$  be a bipartite graph without isolated vertices. Then  $G$  is unmixed if and only if there is a bipartition  $V_1 = \{x_1, \dots, x_g\}$ ,  $V_2 = \{y_1, \dots, y_g\}$  of  $G$  such that: (a)  $\{x_i, y_i\} \in E(G)$  for all  $i$ , and (b) if  $\{x_i, y_j\}$  and  $\{x_j, y_k\}$  are in  $E(G)$  and  $i, j, k$  are distinct, then  $\{x_i, y_k\} \in E(G)$ .*

**Proof.**  $\Rightarrow$ ) Since  $G$  is bipartite, there is a bipartition  $(V_1, V_2)$  of  $G$ , i.e.,  $V(G) = V_1 \cup V_2$ ,  $V_1 \cap V_2 = \emptyset$ , and every edge of  $G$  joins  $V_1$  with  $V_2$ . Let  $g$  be the vertex covering number of  $G$ , i.e.,  $g$  is the number of elements in any minimal vertex cover of  $G$ . Notice that  $V_1$  and  $V_2$  are both minimal vertex covers of  $G$ , hence  $g = |V_1| = |V_2|$ . By König theorem [3, Theorem 10.2, p. 96]  $g$  is the maximum number of independent edges of  $G$ . Therefore after permutation of the vertices we obtain that  $V_1 = \{x_1, \dots, x_g\}$ ,  $V_2 = \{y_1, \dots, y_g\}$ , and that  $\{x_i, y_i\} \in E(G)$  for  $i = 1, \dots, g$ . Thus we have proved that (a) holds. To prove (b) take  $\{x_i, y_j\}$  and  $\{x_j, y_k\}$  in  $E(G)$  such that  $i, j, k$  are distinct. Assume that  $x_i$  is not adjacent to  $y_k$ . Then there is a maximal independent set of vertices  $A$  containing  $x_i$  and  $y_k$ . Notice that  $|A| = g$  because  $G$  is unmixed. Hence  $C = V(G) \setminus A$  is a minimal vertex cover of  $G$  with  $g$  vertices. Since  $x_i$  and  $y_k$  are not on  $C$ , we get that  $y_j$  and  $x_j$  are both in  $C$ . As  $C$  intersects  $\{x_\ell, y_\ell\}$  in at least one vertex for  $\ell \neq j$ , we obtain that  $|C| \geq g + 1$ , a contradiction.

$\Leftarrow$ ) Let  $C$  be a minimal vertex cover of  $G$ . It suffices to prove that  $C$  intersects  $\{x_j, y_j\}$  in exactly one vertex for  $j = 1, \dots, g$ . Assume that  $x_j$  and  $y_j$  belong to  $C$  for some  $j$ . If  $v \in V(G)$ , we denote the neighbor set of  $v$  by  $N_G(v)$ . Thus there are  $x_i \in N_G(y_j) \setminus \{x_j\}$  and  $y_k \in N_G(x_j) \setminus \{y_j\}$  such that  $x_i \notin C$  and  $y_k \notin C$ . Notice that  $i, j, k$  are distinct. Indeed if  $i = k$ , then  $\{x_i, y_i\}$  is an edge of  $G$  not covered by  $C$ , which is impossible. Therefore using (b) we get that  $\{x_i, y_k\}$  is an edge of  $C$ , a contradiction.  $\square$

Ravindra [7] has shown a characterization of well covered bipartite graphs. Namely,  $G$  is well covered if and only if for every edge  $\{x, y\}$  in the perfect matching, the induced subgraph  $\langle N_G(x) \cup N_G(y) \rangle$  is a complete bipartite graph. The advantage of our characterization is that it admits a natural possible extension to hypergraphs and clutters with a perfect matching of König type [5].

As a consequence of Theorem 1.1 we recover the following result on the structure of unmixed trees.

**Corollary 1.2** [8, Theorem 2.4, Corollary 2.5] *Let  $G$  be a tree with at least three vertices. Then  $G$  is unmixed if and only if there is a bipartition  $V_1 = \{x_1, \dots, x_g\}$ ,*

$V_2 = \{y_1, \dots, y_g\}$  of  $G$  such that: (a)  $\{x_i, y_i\} \in E(G)$  for all  $i$ , and (b) for each  $i$  either  $\deg(x_i) = 1$  or  $\deg(y_i) = 1$ .

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